17th International Conference on the Application of Computer
Science and Mathematics in Architecture and Civil Engineering
K. Gürlebeck and C. Könke (eds.)
Weimar, Germany, 12–14 July 2006

CLIFFORD-HERMITE AND TWO-DIMENSIONAL CLIFFORD-GABOR FILTERS FOR EARLY VISION

F. Brackx*, **N. De Schepper, F. Sommen**

Clifford Research Group, Department of Mathematical Analysis
Faculty of Engineering, Ghent University
Galglaan 2, 9000 Gent, Belgium
E-mail: fb@cage.ugent.be

Keywords: Hermite filters, Gabor filters, Clifford analysis.

Abstract. *Among the mathematical models suggested for the receptive field profiles of the human visual system, the Gabor model is well-known and widely used. Another less used model that agrees with the Gaussian derivative model for human vision is the Hermite model, which is based on analysis filters of the Hermite transform. It offers some advantages like being an orthogonal basis and having better match to experimental physiological data.*
In our earlier research both filter models, Gabor and Hermite, have been developed in the framework of Clifford analysis. Clifford analysis offers a direct, elegant and powerful generalization to higher dimension of the theory of holomorphic functions in the complex plane.
In this paper we expose the construction of the Hermite and Gabor filters, both in the classical and in the Clifford analysis framework. We also generalize the concept of complex Gaussian derivative filters to the Clifford analysis setting. Moreover, we present further properties of the Clifford-Gabor filters, such as their relationship with other types of Gabor filters and their localization in the spatial and in the frequency domain formalized by the uncertainty principle.

1 INTRODUCTION

Image processing has been much inspired by the human vision, in particular with regard to early vision. The latter refers to the earliest stage of visual processing responsible for the measurement of local structures such as points, lines, edges and textures in order to facilitate subsequent interpretation of these structures in higher stages (known as high level vision) of the human visual system. This low level visual computation is carried out by cells of the primary visual cortex. The receptive field profiles of these cells can be interpreted as the impulse responses of the cells, which are then considered as filters. According to the Gaussian derivative theory, the receptive field profiles of the human visual system can be approximated quite well by derivatives of Gaussians. Two mathematical models suggested for these receptive field profiles are on the one hand the Gabor model and on the other hand the Hermite model which is based on analysis filters of the Hermite transform. The Hermite filters are derivatives of Gaussians, while Gabor filters, which are defined as harmonic modulations of Gaussians, provide a good approximation to these derivatives. It is important to note that, even if the Gabor model is more widely used than the Hermite model, the latter offers some advantages like being an orthogonal basis and having better match to experimental physiological data.

In our earlier research (see [1, 2]) both filter models, Gabor and Hermite, have been developed in the framework of Clifford analysis. Clifford analysis (see e.g. [3, 4]) offers a direct, elegant and powerful generalization to higher dimension of the theory of holomorphic functions in the complex plane.

In this paper we expose the construction of the Hermite and Gabor filters, both in the classical and in the Clifford analysis framework. We also generalize the concept of complex Gaussian derivative filters to the Clifford analysis setting. Moreover, we present further properties of the Clifford-Gabor filters, such as their relationship with other types of Gabor filters and their localization in the spatial and in the frequency domain formalized by the uncertainty principle.

The outline of the paper is as follows. To make it self-contained, a section on definitions and basic properties of Clifford algebra and Clifford analysis is included (Section 2).

We briefly describe the classical one-dimensional Hermite filters in subsection 3.1, followed by their analogue construction in the Clifford analysis setting. The building blocks for this construction are the so-called radial Clifford-Hermite polynomials which are a multi-dimensional generalization to Clifford analysis of the classical Hermite polynomials on the real line.

Complex Gaussian derivative filters are complex filters consisting of an even real part and an odd imaginary part. The even and odd part are respectively even and odd order derivatives of the Gaussian function (subsection 4.1). In subsection 4.2 we generalize these classical complex Gaussian derivative filters to the framework of Clifford analysis.

The topic of Section 5 is Gabor filters, a prominent tool for local spectral image processing and analysis. Complex Gabor filters (subsection 5.1.1) are closely related to Fourier analysis, since the impulse response of a complex Gabor filter is the conjugated integral kernel of the complex Fourier transform at a certain frequency, multiplied by a Gaussian. Besides complex Gabor filters also real Gabor filters appear in the literature . They are naturally obtained as the real and imaginary part of the complex Gabor filters (subsection 5.1.2). In subsection 5.2 we take a look at two non-classical two-dimensional Gabor filters. First, we briefly describe the so-called quaternionic Gabor filters of Bülow and Sommer, followed by the Clifford-Gabor filters of Ebling and Scheuermann. Next, we proceed with exposing the development of our two-dimensional Clifford-Gabor filters. This new type of Gabor filters arose quite naturally from our study of the so-called Clifford-Fourier transform, a new multi-dimensional Fourier transform in

Clifford analysis. The two-dimensional case of this Clifford-Fourier transform is special in that we succeed in finding a closed form for the kernel of the integral representation. Using the conjugated kernels of the two-dimensional Clifford-Fourier transform modulated by a Gaussian gives rise to the two-dimensional Clifford-Gabor filters (subsection 5.3.1). In subsection 5.3.2 we give an explicit connection between these two-dimensional Clifford-Gabor filters and the standard complex Gabor filters and moreover a surprising connection with the Clifford-Gabor filters of Ebling and Scheuermann. An often cited property of Gabor filters is their optimal simultaneous localization in the spatial and the frequency domain, which is formalized by the uncertainty principle. This property makes Gabor filters suitable for local frequency analysis. In subsection 5.3.3 we show that our two-dimensional Clifford-Gabor filters also exhibit the best possible joint localization in position and frequency space.
In his PhD-thesis Michaelis investigated to which extent one-dimensional derivatives of Gaussians can be approximated by real Gabor filters. In a last section of this paper we briefly discuss the results of his survey.

2 SOME BASIC NOTIONS OF CLIFFORD ALGEBRA AND CLIFFORD ANALYSIS

Clifford algebra may be seen as a generalization to higher dimension of the norm division algebras of the real numbers, the complex numbers and the quaternions; it is a norm algebra where the multiplication is non-commutative, but still associative. Let $\mathbb{R}^m$ be endowed with a non-degenerate quadratic form of signature (p, q), $p + q = m$, and let $(e_1, \ldots, e_m)$ be an orthogonal basis for $\mathbb{R}^{p,q}$. The non-commutative multiplication in the Clifford algebra $\mathbb{R}_{p,q}$, constructed over $\mathbb{R}^{p,q}$, is governed by the rules:

$$\begin{aligned} e_j^2 &= 1 \ , \ j = 1, \ldots, p \\ e_{p+j}^2 &= -1 \ , \ j = 1, \ldots, q \\ e_j e_k + e_k e_j &= 0 \ , \ j \neq k \ , \ j, k = 1, \ldots, m \ . \end{aligned}$$

A canonical basis for $\mathbb{R}_{p,q}$ is obtained by considering for any set $A = \{j_1, \ldots, j_k\} \subset \{1, \ldots, m\} = M$, ordered by $1 \leq j_1 < j_2 < \ldots < j_k \leq m$, the element $e_A = e_{j_1} \ldots e_{j_k}$. Moreover for the empty set $\emptyset$ one puts $e_\emptyset = 1$, the latter being the identity element. Any Clifford number λ in $\mathbb{R}_{p,q}$ may thus be written as $\lambda = \sum_{A \subset M} e_A \lambda_A$, $\lambda_A \in \mathbb{R}$, or still as $\lambda = \sum_{k=0}^{m} [\lambda]_k$, where $[\lambda]_k = \sum_{|A|=k} e_A \lambda_A$ is the so-called k-vector part of λ ($k = 0, 1, \ldots, m$). Denoting by $\mathbb{R}_{p,q}^k$ the subspace of all k-vectors in $\mathbb{R}_{p,q}$, i.e. the image of $\mathbb{R}_{p,q}$ under the projection operator $[\,.\,]_k$, one has the *multi-vector structure* decomposition $\mathbb{R}_{p,q} = \mathbb{R}_{p,q}^0 \oplus \mathbb{R}_{p,q}^1 \oplus \ldots \oplus \mathbb{R}_{p,q}^m$, leading to the identification of $\mathbb{R}$ with the subspace of real scalars $\mathbb{R}_{p,q}^0$ and of $\mathbb{R}^m$ with the subspace of real Clifford vectors $\mathbb{R}_{p,q}^1$. The Clifford number $e_M = e_1 e_2 \ldots e_m$ is mostly called the pseudoscalar; depending on the dimension m, the pseudoscalar commutes or anti-commutes with the k-vectors and squares to ± 1.
Except for subsection 5.2.2, we will only consider the Clifford algebra $\mathbb{R}_{0,m}$ and its complexification $\mathbb{C}_m = \mathbb{C} \otimes \mathbb{R}_{0,m} = \mathbb{R}_{0,m} \oplus i\, \mathbb{R}_{0,m}$. An important automorphism of $\mathbb{C}_m$ leaving the multi-vector structure invariant, is the *Hermitian conjugation* defined by

$$\begin{aligned} (\lambda\mu)^\dagger &= \mu^\dagger \lambda^\dagger \\ (\mu_A e_A)^\dagger &= \mu_A^c e_A^\dagger \quad (A \subset M) \\ e_j^\dagger &= -e_j \quad (j = 1, \ldots, m) \end{aligned}$$

where μ_A^c stands for the complex conjugate of the complex number μ_A.
The Hermitian conjugation leads to a Hermitian inner product and its associated norm on $\mathbb{C}_m$ given by

$$(\lambda, \mu) = [\lambda^\dagger \mu]_0 \ , \ \ |\lambda|^2 = [\lambda^\dagger \lambda]_0 = \sum_A |\lambda_A|^2 \ .$$

The Euclidean space $\mathbb{R}^m$ is embedded in the Clifford algebras $\mathbb{R}_{0,m}$ and $\mathbb{C}_m$ by identifying the point $(x_1, \ldots, x_m)$ with the vector variable $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} e_j x_j \ .$$

The product of two vectors splits up into a scalar part and a 2-vector, also called bivector, part:

$$\underline{x}\,\underline{y} = \underline{x} \cdot \underline{y} \ + \underline{x} \wedge \underline{y} \ ,$$

with

$$\underline{x} \cdot \underline{y} = - < \underline{x}, \underline{y} > = -\sum_{j=1}^{m} x_j y_j$$

and

$$\underline{x} \wedge \underline{y} = \sum_{i=1}^{m} \sum_{j=i+1}^{m} e_i e_j (x_i y_j - x_j y_i) \ .$$

Note that the square of a vector variable $\underline{x}$ is scalar-valued and equals the norm squared up to a minus sign:

$$\underline{x}^2 = - < \underline{x}, \underline{x} > = -|\underline{x}|^2 \ .$$

The elliptic, rotation-invariant, vector differential operator of the first order

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j} \ ,$$

called *Dirac operator*, may be looked upon as the "square root" of the Laplace operator in $\mathbb{R}^m$:

$$\Delta_m = -\partial_{\underline{x}}^2 \ . \tag{1}$$

This factorization of the Laplace operator is one of the most fundamental features in Clifford analysis. It is precisely this Dirac operator which underlies the notion of monogenicity of a function, a notion which is the multi-dimensional counterpart to that of holomorphy in the complex plane. A function f defined and differentiable in an open region Ω of $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$ or $\mathbb{C}_m$, is called left-monogenic in Ω if $\partial_{\underline{x}} f = 0$.
Note that by the factorization (1) of the Laplace operator, a special relationship is established between monogenic functions and harmonic functions of several variables, in that the properties of monogenic functions constitute a refinement of those of harmonic functions.

In Clifford analysis extensive use is made of the standard tensorial multi-dimensional *Fourier transform*

$$\mathcal{F}[f](\underline{y}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} \exp\left(-i < \underline{y}, \underline{x} >\right) f(\underline{x}) \, dV(\underline{x}) \ , \tag{2}$$

where $dV(\underline{x})$ stands for for the Lebesgue measure on $\mathbb{R}^m$.
This Fourier transform satisfies:

- the *multiplication rule*:

$$\mathcal{F}[\underline{x} f(\underline{x})](\underline{y}) = i\partial_{\underline{y}} \mathcal{F}[f(\underline{x})](\underline{y})$$

- the *differentiation rule*:

$$\mathcal{F}[\partial_{\underline{x}} f(\underline{x})](\underline{y}) = i\underline{y} \mathcal{F}[f(\underline{x})](\underline{y}) \ . \tag{3}$$

Moreover, it is an isometry on the space of square integrable functions, in other words, for all $f, g \in L_2(\mathbb{R}^m, dV(\underline{x}))$ the *Parseval formula* holds:

$$< f, g > = < \mathcal{F}[f], \mathcal{F}[g] >$$

with the inner product given by

$$< f, g > = \int_{\mathbb{R}^m} f^\dagger(\underline{x})\ g(\underline{x})\ dV(\underline{x}) \ .$$

In particular, for each $f \in L_2(\mathbb{R}^m, dV(\underline{x}))$ one has:

$$\|f\|_2 = \|\mathcal{F}[f]\|_2 \tag{4}$$

where the L_2-norm is defined as

$$\begin{aligned} \|f\|_2^2 &= [< f, f >]_0 \\ &= \int_{\mathbb{R}^m} |f(\underline{x})|^2\ dV(\underline{x}) \ . \end{aligned}$$

3 HERMITE FILTERS

3.1 The classical one-dimensional Hermite filters

The Hermite transform was introduced in [5] as a signal expansion technique in which a signal is windowed by a Gaussian at equidistant positions and is locally described by a weighted sum of polynomials. It has been used for a.o. local analysis of images (generally to process edges), image coding, image deblurring, noise reduction and estimation of perceived noise and blur.

The Hermite transform first localizes the original signal $f(x)$ by multiplying it by a Gaussian window function

$$V(x) = \frac{1}{\sqrt{\sqrt{\pi}\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \ .$$

In order to have a complete description of the signal $f(x)$, the localization process should be repeated at a sufficient number of window positions, the spacing between the windows being chosen equidistant. In this way the following expansion of the original signal $f(x)$ is obtained

$$f(x) = \frac{1}{W(x)} \sum_{k=-\infty}^{+\infty} f(x) V(x - kT) \tag{5}$$

where

$$W(x) = \sum_{k=-\infty}^{+\infty} V(x - kT)$$

is the so-called *weight function*.

The next step in the Hermite transform is the decomposition of the localized signal $f(x)V(x-kT)$ into the orthonormal functions

$$K_n(x) = \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{x}{\sigma}\right) \, V(x) \ , \quad n = 0, 1, 2, \ldots$$

where H_n is the standard *Hermite polynomial* of order n ($n = 0, 1, 2, \ldots$) defined by the so-called *Rodrigues formula*

$$H_n(x) = (-1)^n \, \exp\left(x^2\right) \frac{d^n}{dx^n}\big(\exp\left(-x^2\right)\big) \ . \tag{6}$$

Under very general conditions for the original signal $f(x)$, we get the following decomposition of the localized signal

$$V(x-kT)f(x) = \sum_{n=0}^{\infty} c_n(kT) \, K_n(x-kT) \tag{7}$$

where the so-called *Hermite coefficients* $c_n(kT)$ are obtained by convolving the signal with the so-called *Hermite filters* d_n^σ, i.e.

$$c_n(kT) = (f * d_n^\sigma)(kT) = \int_{-\infty}^{+\infty} f(x) \, d_n^\sigma(kT - x) \, dx$$

with

$$d_n^\sigma(x) = \frac{(-1)^n}{\sqrt{2^n n!}} H_n\left(\frac{x}{\sigma}\right) \, (V(x))^2 \ .$$

By means of the Rodrigues formula (6) for the Hermite polynomials, the Hermite filters can be rewritten as Gaussian derivatives:

$$d_n^\sigma(x) = \frac{\sigma^n}{\sqrt{2^n n!}} \, \frac{d^n}{dx^n}\big(V(x)^2\big) \ .$$

Hence, as explained in the introduction, the Hermite transform models the information analysis carried out by the cortical visual receptive fields. Because these receptive fields occur in varying size, each field is suited for detecting the presence of a specific spatial frequency. With the Hermite transform, field sizes can be modelled by varying the standard deviation σ of the Gaussian envelope, while orientation selectivity can be obtained by rotation of the Hermite filters.
Note that the Fourier transform of the Hermite filter d_n^σ takes the form of a Gaussian modulated by a monomial of degree n :

$$\mathcal{F}[d_n^\sigma](y) = \frac{\sigma^n}{\sqrt{2^n n!}} (iy)^n \, \exp\left(-\frac{\sigma^2 y^2}{4}\right) \ .$$

Combining (5) and (7), we get the following expansion of the complete signal:

$$f(x) = \frac{1}{W(x)} \sum_{n=0}^{\infty} \sum_{k=-\infty}^{+\infty} c_n(kT) \, K_n(x-kT) \ .$$

The Hermite transform is generalized to higher dimension in a tensorial manner.

Remark 3.1 *The Hermite transform provides the connection between the derivatives of Gaussians and the so-called* Hermite functions. *In the Hermite transform, the analysis functions d_n^σ are the derivatives of Gaussians, whereas the reconstruction functions K_n are the Hermite functions. The difference between the Hermite function and the derivative of the Gaussian of order n is the scale of the Gaussian in relation to the scale of the Hermite polynomial. In case of the Hermite functions the Hermite polynomials grow as fast as the exponential decays and hence the maxima of the Hermite functions have all about the same height, giving them the shape of a truncated sine/cosine wave.*
The Hermite functions have two interesting properties. First, they maximize the uncertainty principle (see subsection 5.3.3) and second, as for the Gaussian, their Fourier transform has the same functional form as the function itself.

3.2 The Clifford-Hermite filters

In [1] a new multi-dimensional Hermite transform was developed in Clifford analysis using the so-called *radial Clifford-Hermite polynomials* of Clifford analysis; we call it the *Clifford-Hermite transform*. These radial Clifford-Hermite polynomials were introduced by Sommen in [6] as a multi-dimensional generalization to Clifford analysis of the classical Hermite polynomials on the real line. They are defined by the Rodrigues formula

$$H_n(\underline{x}) = (-1)^n \exp\left(\frac{|\underline{x}|^2}{2}\right) \partial_{\underline{x}}^n \left(\exp\left(-\frac{|\underline{x}|^2}{2}\right)\right) \quad , \quad n = 0, 1, 2, \ldots \tag{8}$$

and are orthogonal on $\mathbb{R}^m$ with respect to the exponential weight function $\exp\left(-\frac{|\underline{x}|^2}{2}\right)$. Moreover $H_n(\underline{x})$ turns out to be a polynomial of degree n in the variable $\underline{x}$ with real coefficients. Furthermore, $H_{2n}(\underline{x})$ only contains even powers of $\underline{x}$, while $H_{2n+1}(\underline{x})$ only contains odd ones. Hence the radial Clifford-Hermite polynomials are alternatively scalar- or vector-valued.

First, using the Gaussian window function

$$V(\underline{x}) = \exp\left(-\frac{|\underline{x}|^2}{2\sigma^2}\right) \quad ,$$

we get the following decomposition of the original real-valued signal $f(\underline{x})$:

$$f(\underline{x}) = \frac{1}{W(\underline{x})} \sum_{\underline{p} \in P} f(\underline{x}) V(\underline{x} - \underline{p}) \tag{9}$$

with $W(\underline{x}) = \sum_{\underline{p} \in P} V(\underline{x} - \underline{p})$ the positive weight function and P a sampling grid in $\mathbb{R}^m$.

Next, under very general conditions for the original signal $f(\underline{x})$, we can decompose the localized signal $V(\underline{x}-\underline{p})f(\underline{x})$ into the orthogonal functions $K_n(\underline{x}) = H_n\left(\frac{\sqrt{2}\underline{x}}{\sigma}\right) V(\underline{x})$ as follows:

$$V(\underline{x} - \underline{p}) f(\underline{x}) = \sum_{j=0}^{\infty} c_j(\underline{p}) \; K_j^{\dagger}(\underline{x} - \underline{p}) \quad . \tag{10}$$

The expansion coefficients $c_n(\underline{p})$ are called the *Clifford-Hermite coefficients* and may be expressed as the convolution of the signal $f(\underline{x})$ with the *Clifford-Hermite filter functions* d_n^σ :

$$c_n(\underline{p}) = (f * d_n^\sigma)(\underline{p})$$

with

$$
\begin{aligned}
d_n^\sigma(\underline{x}) &= \frac{(-1)^n 2^{m/2}}{\sigma^m \gamma_n} H_n\left(\frac{\sqrt{2}\underline{x}}{\sigma}\right) \left(V(\underline{x})\right)^2 \\
&= \frac{2^{(m-n)/2}\sigma^{n-m}}{\gamma_n}\, \partial_{\underline{x}}^n \left(V(\underline{x})^2\right) \ . \qquad (11)
\end{aligned}
$$

Here γ_n denotes a real constant depending on the parity of n. The second expression in (11) is obtained by using the Rodrigues formula (8). Note that the parameters of the alternatively scalar- or vector-valued Clifford-Hermite filters are the scale σ of the Gaussian and the derivative order n. Moreover, their Fourier transform in spherical co-ordinates $\underline{y} = \rho\underline{\xi}$, $\rho = |\underline{y}|$, $\underline{\xi} \in S^{m-1}$ with S^{m-1} the unit sphere in $\mathbb{R}^m$, is given by

$$
\mathcal{F}[d_n^\sigma](\underline{y}) = \frac{(i\sigma)^n}{\gamma_n 2^{n/2}}\, \underline{\xi}^n \rho^n \exp\left(-\frac{\sigma^2\rho^2}{4}\right) \ .
$$

Hence the Clifford-Hermite filters are polar separable, i.e. their Fourier transform is expressed as the product of a spatial frequency tuning function and an orientation tuning function.

Finally, combining the formulae (9) and (10), we obtain the following decomposition of the complete signal:

$$
f(\underline{x}) = \frac{1}{W(\underline{x})} \sum_{j=0}^{\infty} \sum_{\underline{p}\in P} c_j(\underline{p}) K_j^\dagger(\underline{x} - \underline{p}) \ .
$$

Remark 3.2

1. The Clifford-Hermite transform introduced above, was also constructed in the framework of quaternionic analysis (see [7]).

2. In [1] the Clifford-Hermite transform was generalized by using the so-called generalized Clifford-Hermite polynomials of Clifford analysis, also introduced in [6].

3. In [1] the similarities and differences between the Clifford-Hermite transform and the so-called Clifford-Hermite Continuous Wavelet Transform were investigated.

4 COMPLEX GAUSSIAN DERIVATIVE FILTERS

4.1 The classical one-dimensional setting

Derivatives of Gaussians are motivated by, among others, their optimal feature detection properties and their close connection to differential geometrical methods and Taylor expansions. Moreover, they have a simple analytical structure what makes them easy to handle.
The *Gaussian* has the general form

$$
D_0(x) = N \exp(-ax^2)
$$

with N a normalization constant depending on the parameter a. For this normalization constant there are two possibilities:

- *amplitude normalization* of the Gaussian:

$$
\|D_0\|_1 = \int_{-\infty}^{+\infty} D_0(x)\, dx = 1 \ ,
$$

which yields

$$N_1 = \sqrt{\frac{a}{\pi}}$$

- *energy normalization* of the Gaussian:

$$\|D_0\|_2 = \left(\int_{-\infty}^{+\infty} (D_0(x))^2 \, dx \right)^{1/2} = 1 \ ,$$

which implies

$$N_2 = \left(\frac{2a}{\pi} \right)^{1/4} \ .$$

There are several possibilities for the coefficient a in the exponent:

- the standard form is $a = \frac{1}{2\sigma^2}$, where σ is the standard deviation
- in the Hermite transform (see Section 3) a Gaussian window as well as the squared window are used and hence a can be $a = \frac{1}{\sigma^2}$.

The *derivatives of the Gaussian* are defined as follows:

$$\begin{aligned} D_n(x) &= \frac{d^n}{dx^n} (D_0(x)) \\ &= (-1)^n N(\sqrt{a})^n H_n(\sqrt{a}x) \exp(-ax^2) \ , \quad n = 0, 1, 2, \ldots \end{aligned} \tag{12}$$

with H_n the classical Hermite polynomial of order n defined by (6).
Their Fourier transform is given by

$$\mathcal{F}[D_n](y) = (iy)^n \mathcal{F}[D_0](y) = N \frac{1}{\sqrt{2a}} (iy)^n \exp\left(-\frac{y^2}{4a} \right) \ .$$

There is growing evidence that feature detection has to be done by energy detectors that consist of complex filters of an even real part and an odd imaginary part.
In [8] complex odd/even filters from Gaussian derivatives with $a = \frac{1}{2\sigma^2}$ are considered. The dependence of the L_1- and L_2-norm of D_n on the order n and the amplitude is given by σ^{-n}, i.e.

$$\|D_n\|_1 \approx \sigma^{-n} \quad \text{and} \quad \|D_n\|_2 \approx \sigma^{-n} \ .$$

Hence, to make the relative strength of the odd and even part of the complex filter independent of their scale, the lower order function has to be multiplied by a factor σ^{-1}. Moreover, a free parameter k at the imaginary odd part is introduced which controls the relative weight of the odd and even part.
The above reasoning yields the so-called *complex Gaussian derivative filters*:

$$F(x) := D_{2s}(x) \pm ik\sigma^{\pm 1} D_{2s\pm 1}(x) \ .$$

The relative sign of the odd and even part is chosen such that the Fourier energy at negative frequencies compensates.
In [8] several possible choices for the free parameter k are discussed, according to the following criteria:

1. same L_2- or L_1-norm of the odd and even part
2. minimal Fourier energy at negative frequencies of the complex filter F
3. monomodal energy of the complex filter F
4. fit of the energy of F to a Gaussian
5. linear phase of the complex filter F.

4.2 The Clifford analysis setting

In this subsection we generalize the classical complex Gaussian derivative filters to the framework of Clifford analysis.
Starting point is the *Gaussian*

$$D_0(\underline{x}) = N \exp\left(-a|\underline{x}|^2\right) \ ,$$

where, similar to the classical case, the coefficient a in general takes the form $a = \dfrac{1}{2\sigma^2}$ or, in case of the Hermite transform, $a = \dfrac{1}{\sigma^2}$.
The normalization constant N is chosen according to

- *amplitude normalization* of the Gaussian:

  $$\|D_0\|_1 = \int_{\mathbb{R}^m} |D_0(\underline{x})| \, dV(\underline{x}) = N_1 \int_{\mathbb{R}^m} \exp\left(-a|\underline{x}|^2\right) dV(\underline{x}) = 1 \ ,$$

  from which we obtain

  $$N_1 = \left(\frac{a}{\pi}\right)^{m/2}$$

- *energy normalization* of the Gaussian:

  $$\|D_0\|_2 = \left(\int_{\mathbb{R}^m} |D_0(\underline{x})|^2 \, dV(\underline{x})\right)^{1/2} = N_2 \left(\int_{\mathbb{R}^m} \exp\left(-2a|\underline{x}|^2\right) dV(\underline{x})\right)^{1/2} = 1 \ ,$$

  which yields

  $$N_2 = \left(\frac{2a}{\pi}\right)^{m/4} \ .$$

Naturally, we define the *derivatives of the Gaussian* in the Clifford analysis setting by means of the Dirac operator:

$$D_n(\underline{x}) = \partial_{\underline{x}}^n \left(D_0(\underline{x})\right) \ , \quad n = 0, 1, 2, \ldots$$

Using the Rodrigues formula (8), they can be expressed in terms of the radial Clifford-Hermite polynomials:

$$D_n(\underline{x}) = (-1)^n N (\sqrt{2a})^n H_n(\sqrt{2a}\underline{x}) \exp\left(-a|\underline{x}|^2\right) \ .$$

Note that a derivative of even order is an even function of the vector variable $\underline{x}$, while a derivative of odd order is an odd function. Moreover, these derivatives of the Gaussian are alternatively

scalar- or vector-valued.
Using the differentiation rule (3) for the Fourier transform, we easily obtain

$$\begin{aligned}\mathcal{F}[D_n](\underline{y}) &= (i\underline{y})^n \mathcal{F}[D_0](\underline{y}) \\ &= N\left(\frac{1}{\sqrt{2a}}\right)^m (i\underline{y})^n \exp\left(-\frac{|\underline{y}|^2}{4a}\right) \ .\end{aligned}$$

Let us now calculate the L_2-norm of the Gaussian derivatives. This is most easily done in frequency space. By means of the Parseval formula (4) for the Fourier transform, we have

$$\begin{aligned}\|D_n\|_2 &= \|\mathcal{F}[D_n]\|_2 \\ &= \left(\int_{\mathbb{R}^m} |\mathcal{F}[D_n](\underline{y})|^2 \, dV(\underline{y})\right)^{1/2} \ .\end{aligned}$$

As

$$\begin{aligned}|\mathcal{F}[D_n](\underline{y})|^2 &= \left[\left(\mathcal{F}[D_n](\underline{y})\right)^\dagger \ \mathcal{F}[D_n](\underline{y})\right]_0 \\ &= N^2\left(\frac{1}{2a}\right)^m \exp\left(-\frac{|\underline{y}|^2}{2a}\right) |\underline{y}|^{2n} \ ,\end{aligned}$$

the L_2-norm of D_n becomes

$$\|D_n\|_2 = N\left(\frac{1}{2a}\right)^{m/2} \left(\int_{\mathbb{R}^m} \exp\left(-\frac{|\underline{y}|^2}{2a}\right) |\underline{y}|^{2n} \, dV(\underline{y})\right)^{1/2} \ . \tag{13}$$

Using spherical co-ordinates $\underline{y} = \rho\underline{\xi}$, $\rho = |\underline{y}|$ and $\underline{\xi} \in S^{m-1}$, expression (13) becomes

$$\begin{aligned}\|D_n\|_2 &= N\left(\frac{1}{2a}\right)^{m/2} \left(\int_0^{+\infty}\int_{S^{m-1}} \exp\left(-\frac{\rho^2}{2a}\right)\rho^{2n+m-1} dS(\underline{\xi})\, d\rho\right)^{1/2} \\ &= N\left(\frac{1}{2a}\right)^{m/2} A_m^{1/2} \left(\int_0^{+\infty} \exp\left(-\frac{\rho^2}{2a}\right)\rho^{2n+m-1} d\rho\right)^{1/2}\end{aligned}$$

where A_m denotes the area of the unit sphere S^{m-1} in $\mathbb{R}^m$.
By means of the following integral ($\alpha > 0$) [9]

$$\int_0^{+\infty} x^\ell \exp\left(-\alpha x^2\right) dx = \begin{cases} \frac{1.3\ldots(2k-1)\sqrt{\pi}}{2^{k+1}\alpha^{k+1/2}} & \text{for } \ell = 2k \\ \frac{k!}{2\alpha^{k+1}} & \text{for } \ell = 2k+1 \end{cases} \ ,$$

we finally obtain

$$\|D_n\|_2 = \begin{cases} N\sqrt{A_m \left(n-1+\frac{m}{2}\right)! \, 2^{n-m/2-1} a^{n-m/2}} & \text{if } m \text{ is even} \\ N\sqrt{\frac{A_m 1.3\ldots(2n+m-2)\sqrt{\pi}a^{n-m/2}}{2^{m+1/2}}} & \text{if } m \text{ is odd.} \end{cases}$$

For the standard case where $a = \frac{1}{2\sigma^2}$, this becomes

$$\|D_n\|_2 = \begin{cases} N\sqrt{\frac{A_m\left(n-1+\frac{m}{2}\right)! \ \sigma^{m-2n}}{2}} & \text{if } m \text{ is even} \\ N\sqrt{\frac{A_m 1.3\ldots(2n+m-2)\sqrt{\pi}\sigma^{m-2n}}{2^{n+m/2+1/2}}} & \text{if } m \text{ is odd.} \end{cases} \tag{14}$$

Now we have all the necessary tools to define complex Gaussian derivative filters in the Clifford analysis setting.
We consider Gaussian derivatives D_n with $a = \frac{1}{2\sigma^2}$. From (14) we see that the dependence of the L_2-norm of D_n on the order n and the amplitude is given by σ^{-n}. Hence, again the lower order function has to be multiplied by a factor σ^{-1} in order to make the relative strength of the odd and even part of the complex filter independent of their scale. Thus, similar to the classical case, we define the *complex Gaussian derivative filters in Clifford analysis* as follows:

$$F(\underline{x}) := D_{2s}(\underline{x}) \pm ik\sigma^{\pm 1} D_{2s\pm 1}(\underline{x}) \ .$$

In the classical case, there are several choices for the free parameter k in the complex filter (see subsection 4.1). In this Clifford analysis setting, we restrict ourselves to the criterion of equal L_2-norm of the odd and even part. Hence, we search for the value of k such that the even and odd part of the complex filter have the same L_2-norm; this value of k is denoted by k_{L_2}.
Let us first consider the case where the dimension m is even. From (14), we have

$$\frac{\|D_n\|_2}{\|D_{n-1}\|_2} = \sqrt{n-1+\frac{m}{2}}\,\frac{1}{\sigma} \ .$$

Let n denote the higher order of the odd and even part of the filter.
If $n = 2s$ is even, which means that we consider the complex filter

$$F(\underline{x}) = D_{2s}(\underline{x}) - ik\sigma^{-1} D_{2s-1}(\underline{x}) \ ,$$

we obtain

$$k_{L_2} = \sqrt{n-1+\frac{m}{2}} \ .$$

Similarly, if $n = 2s+1$ is odd, which implies that the complex filter takes the form

$$F(\underline{x}) = D_{2s}(\underline{x}) + ik\sigma D_{2s+1}(\underline{x}) \ ,$$

we derive

$$k_{L_2} = \frac{1}{\sqrt{n-1+\frac{m}{2}}} \ .$$

The case where the dimension m is odd yields the same results, since we then also have that

$$\frac{\|D_n\|_2}{\|D_{n-1}\|_2} = \sqrt{n-1+\frac{m}{2}}\,\frac{1}{\sigma} \ .$$

5 GABOR FILTERS

5.1 The classical one-dimensional Gabor filters

One of the most prominent tools for local spectral image processing and analysis are Gabor filters. They were first introduced in the field of one-dimensional signal processing by Gabor in [10] for a joint time-frequency analysis. Gabor filters have the main advantage of being simultaneously optimally localized in the spatial and in the frequency domain. Hence spatial and frequency properties are optimally analyzed at the same time by Gabor filters (see subsection 5.3.3). Gabor filters also give access to the local phase of a signal. It has been shown that there is a close correspondence between the local structure of a signal and its local phase. Furthermore, certain regions in the human visual cortex can be modelled as Gabor filters (see Section 1). Hence, Gabor filters conform well to the human visual system's capabilities.

Gabor filters have been successfully applied to different image processing and analysis tasks such as texture segmentation (see for e.g. [11, 12]), edge detection and local phase and frequency estimation for image matching.

5.1.1 One-dimensional complex Gabor filters

In this section we consider the classical Fourier transform with the factor 2π in the exponential:

$$\mathcal{F}_{cl}[f](u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\ \exp\left(-i2\pi ux\right) dx \ .$$

Complex Gabor filters are closely related to Fourier analysis in the following way. They are linear shift invariant (abbreviated LSI) filters, hence they can be applied by simply convolving the signal with the impulse response of the filter. The *impulse response* h of a complex Gabor filter is the complex conjugated integral kernel of the classical Fourier transform $\mathcal{F}_{cl}$ of some frequency u^* multiplied with a Gaussian g centered at the origin, i.e.

$$h(x) = g(x) \exp\left(i2\pi u^* x\right)$$

with

$$g(x) = N \exp\left(-\frac{x^2}{2\sigma^2}\right) \ . \tag{15}$$

While Gabor filters analyze the local spectral properties of a signal, the Fourier transform decomposes a signal into its global spectral components. Hence the Fourier transform can be considered the basis upon which Gabor filters were introduced.

The parameters of the complex Gabor filter are the *normalization constant* N, the *center frequency* u^* and the *variance* or *standard deviation* σ of the Gaussian. Normally, N is chosen such that the Gaussian is amplitude normalized, i.e.

$$\|g(x)\|_1 = \int_{-\infty}^{+\infty} g(x)\ dx = 1 \ ,$$

which implies

$$N = \frac{1}{\sqrt{2\pi}\sigma} \ .$$

Sometimes other parameterizations of the complex Gabor filter than the one given above are used, viz.

$$\begin{aligned} h(x) &= g(x) \exp\left(iy^* x\right) \\ &= g(x) \exp\left(\frac{icx}{\sigma}\right) \ . \end{aligned}$$

Here $y^* = 2\pi u^*$ is the *angular frequency* and $c = y^* \sigma$ is the *oscillation parameter*.

The *transfer function* of a complex Gabor filter is a shifted Gaussian:

$$\begin{aligned} H(u) &:= \mathcal{F}_{cl}[h](u) \\ &= \frac{1}{\sqrt{2\pi}} \exp\left(-2\pi^2\sigma^2(u-u^*)^2\right) \ . \end{aligned}$$

In terms of the angular frequency $y = 2\pi u$ this becomes

$$H(y) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\sigma^2}{2}(y-y^*)^2\right) \ .$$

Hence, Gabor filters are *bandpass* filters. The main amount of energy of the Gabor filter is centered around the frequency u^* in the positive half of the frequency domain.

Analogously, the definition of the two-dimensional complex Gabor filters is based on the classical two-dimensional Fourier transform.

5.1.2 One-dimensional real Gabor filters

Besides complex Gabor filters, also real Gabor filters appear in the literature (see e.g. [13, 14]). Naturally they are obtained as the real and imaginary part of the complex Gabor filters introduced above. Hence the *impulse responses* of these real Gabor filters take the form

$$h_c(x) = g(x)\cos(2\pi u^* x) = g(x)\cos(y^* x) = g(x)\cos\left(\frac{cx}{\sigma}\right)$$

and

$$h_s(x) = g(x)\sin(2\pi u^* x) = g(x)\sin(y^* x) = g(x)\sin\left(\frac{cx}{\sigma}\right)$$

with g the Gaussian given by (15).
Their associated *transfer functions* are

$$\begin{aligned} H_{c/s}(u) &:= \mathcal{F}_{cl}[h_{c/s}](u) \\ &= N\,\frac{\sigma}{2}\Big[\exp\left(-\sigma^2 2\pi^2(u+u^*)^2\right) \pm \exp\left(-\sigma^2 2\pi^2(u-u^*)^2\right)\Big] \end{aligned}$$

where the plus sign, respectively the minus sign, corresponds with the cosine, respectively the sine, Gabor filter. The above expression can be rewritten in terms of the angular frequency:

$$H_{c/s}(y) = N\frac{\sigma}{2}\left[\exp\left(-\sigma^2\frac{(y+y^*)^2}{2}\right) \pm \exp\left(-\sigma^2\frac{(y-y^*)^2}{2}\right)\right] \ .$$

5.2 Different types of two-dimensional Gabor filters

5.2.1 Quaternionic Gabor filters

In [15] Bülow and Sommer define a so-called *quaternionic Fourier transform* of two-dimensional signals $f(x_1, x_2)$ taking their values in the algebra $\mathbb{H}$ of real quaternions. If, traditionally, the basis vectors in $\mathbb{H}$ are denoted by i and j, with $i^2 = j^2 = -1$, then this quaternionic Fourier transform takes the form

$$\mathcal{F}^q[f](u_1, u_2) = \int_{\mathbb{R}^2} \exp(-2\pi i u_1 x_1)\ f(x_1, x_2)\ \exp(-2\pi j u_2 x_2)\ dx_1 dx_2 \ .$$

Furthermore, in [11, 16] Bülow and Sommer construct so-called *quaternionic Gabor filters* and apply them to the problems of disparity estimation and texture segmentation. The impulse response h^q of a quaternionic Gabor filter is a Gaussian windowed kernel function of the quaternionic Fourier transform:

$$h^q(\underline{x}) = g(\underline{x})\exp(i 2\pi u_1^* x_1)\ \exp(j 2\pi u_2^* x_2)$$

with

$$g(\underline{x}) = N \exp\left(-\frac{x_1^2 + (\epsilon x_2)^2}{2\sigma^2}\right) \quad . \tag{16}$$

The parameter ϵ is the so-called *aspect ratio*.
In the quaternionic frequency domain, these Gabor filters are shifted Gaussians:

$$\begin{aligned} H^q(\underline{u}) &:= \mathcal{F}^q[h^q](\underline{u}) \\ &= \exp\left(-2\pi^2\sigma^2\left((u_1 - u_1^*)^2 + \frac{(u_2 - u_2^*)^2}{\epsilon^2}\right)\right) \quad . \end{aligned}$$

5.2.2 Gabor filters of Ebling and Scheuermann

In [17] Ebling and Scheuermann study the Clifford-Fourier transformation of two- and three-dimensional signals, using the respective Fourier kernels $\exp\big(-2\pi e_{12}(u_1x_1 + u_2x_2)\big)$ and $\exp\big(-2\pi e_{123}(u_1x_1 + u_2x_2 + u_3x_3)\big)$, where e_{12} and e_{123} are the pseudoscalars in the Clifford algebras $\mathbb{R}_{2,0}$ and $\mathbb{R}_{3,0}$ respectively. Moreover, in [18] they introduce two- and three-dimensional Gabor filters based on these Clifford-Fourier transforms and use them for the description of local patterns in flow fields. The impulse responses h^e of their two-dimensional Gabor filters take the form

$$\begin{aligned} h^e(\underline{x}) &= g(\underline{x}')\ \exp\left(2\pi e_{12}(u_1^*x_1 + u_2^*x_2)\right) \\ &= g(\underline{x}')\ \exp\left(e_{12}(y_1^*x_1 + y_2^*x_2)\right) \end{aligned}$$

with $\underline{x}'$ a rotated version of $\underline{x}$ and g the Gaussian given by (16).

5.3 The two-dimensional Clifford-Gabor filters

5.3.1 Definition

In [19] a new multi-dimensional Fourier transform in the framework of Clifford analysis, the so-called *Clifford-Fourier transform*, is introduced. The idea behind its definition originates from an alternative representation for the standard tensorial multi-dimensional Fourier transform given by (2). It is indeed so that this classical Fourier transform can be seen as the operator exponential

$$\mathcal{F} = \exp\left(-i\frac{\pi}{2}\mathcal{H}\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(-i\frac{\pi}{2}\right)^k \mathcal{H}^k$$

where $\mathcal{H}$ is the scalar-valued differential operator

$$\mathcal{H} = \frac{1}{2}(-\Delta_m + r^2 - m) \quad .$$

Note that due to the scalar character of the standard Fourier kernel, the Fourier spectrum inherits its Clifford algebra character from the original signal, without any interaction with the Fourier kernel. So in order to genuinely introduce the Clifford analysis character in the Fourier transform, the idea occurred to us to replace the scalar-valued operator $\mathcal{H}$ in the operator exponential by a Clifford algebra-valued one. To that end we aimed at factorizing the operator $\mathcal{H}$, making use of the factorization of the Laplace operator by the Dirac operator. Splitting $\mathcal{H}$ into a sum of Clifford algebra-valued second order operators, leads in a natural way to a *pair* of transforms $\mathcal{F}_{\mathcal{H}^\pm}$, the harmonic average of which is precisely the standard Fourier transform $\mathcal{F}$.

So one could say that the Clifford-Fourier transform $\mathcal{F}_{\mathcal{H}^\pm}$ offers a refinement of the classical Fourier transform $\mathcal{F}$ in a similar way as monogenic functions constitute a refinement of harmonic ones.

The two-dimensional case of the Clifford-Fourier transform is special in that we are able to find a closed form for the kernel of the integral representation. Indeed, the two-dimensional Clifford-Fourier transform takes the form

$$\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{y}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp\left(\pm(\underline{y} \wedge \underline{x})\right) f(\underline{x})\, dV(\underline{x}) \ .$$

This closed form enables us to generalize the well-known results for the standard Fourier transform both in the L_1 and in the L_2 context (see [2]). Note that we have not succeeded yet in obtaining such a closed form in arbitrary dimension.

Using the conjugated kernels of the two-dimensional Clifford-Fourier transform $\mathcal{F}_{\mathcal{H}^\pm}$ modulated by a Gaussian, a new type of two-dimensional Gabor filters is defined in [2]. These so-called *two-dimensional Clifford-Gabor filters* $\mathcal{G}^\pm$ are LSI filters with impulse response given by

$$\begin{aligned} h^\pm(\underline{x}) &= g(\underline{x})\ \exp\left(\pm(\underline{x} \wedge \underline{y}^*)\right) \qquad (17) \\ &= g(\underline{x})\ \cos\left(x_1 y_2^* - x_2 y_1^*\right) \pm e_{12}\ g(\underline{x})\ \sin\left(x_1 y_2^* - x_2 y_1^*\right) \ , \end{aligned}$$

where g is the Gaussian given by

$$g(\underline{x}) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{|\underline{x}|^2}{2\sigma^2}\right) \ .$$

Note that these Clifford-Gabor filters consist of a scalar and a bivector part, i.e. they are so-called parabivectors. Their parameters are the angular frequency $\underline{y}^*$ and the variance σ which determines the scale of the Gaussian envelope.
It turns out that both types of Clifford-Gabor filters $\mathcal{G}^\pm$ have the same transfer function:

$$\begin{aligned} H^\pm(\underline{y}) &:= \mathcal{F}_{\mathcal{H}^\pm}[h^\pm](\underline{y}) \\ &= \frac{1}{2\pi} \exp\left(-\frac{\sigma^2}{2}|\underline{y} - \underline{y}^*|^2\right) \ . \end{aligned}$$

Note that, similar to the classical case, the transfer functions are shifted Gaussians, which implies that the Clifford-Gabor filters are bandpass filters.

5.3.2 Relationship with other Gabor filters

In this subsection we first derive an explicit connection between the two-dimensional Clifford-Gabor filters $\mathcal{G}^\pm$ and the classical complex Gabor filters (see subsection 5.1.1). In this derivation the following Clifford numbers play a crucial role:

$$P^\pm = \frac{1}{2}(1 \pm i e_{12}) \ .$$

They are self-adjoint mutually orthogonal idempotents which, by multiplication, transform e_{12} into the imaginary unit i. More precisely, they show the following properties:

$$P^+ + P^- = 1 \ , \ \ P^+P^- = P^-P^+ = 0 \ , \ \ (P^\pm)^2 = P^\pm \qquad (18)$$

and

$$P^+ i = P^+(-e_{12}) = (-e_{12})P^+ \quad , \quad P^- i = P^- e_{12} = e_{12} P^- \quad . \tag{19}$$

Using property (18) we can rewrite the impulse response h^+ of the Clifford-Gabor filter $\mathcal{G}^+$ as

$$\begin{aligned} h^+(\underline{x}) &= g(\underline{x}) \ \exp(\underline{x} \wedge \underline{y}^*) \\ &= g(\underline{x}) \ P^+ \exp(e_{12}(x_1 y_2^* - x_2 y_1^*)) + g(\underline{x}) \ P^- \exp(e_{12}(x_1 y_2^* - x_2 y_1^*)) \quad . \end{aligned} \tag{20}$$

Furthermore, by means of (19) we obtain consecutively

$$\begin{aligned} P^+ \exp(e_{12}(x_1 y_2^* - x_2 y_1^*)) &= P^+ \sum_{k=0}^{\infty} \frac{(e_{12})^k}{k!} (x_1 y_2^* - x_2 y_1^*)^k \\ &= P^+ \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} (x_1 y_2^* - x_2 y_1^*)^k \\ &= P^+ \exp(-i(x_1 y_2^* - x_2 y_1^*)) \end{aligned}$$

and similarly

$$P^- \exp(e_{12}(x_1 y_2^* - x_2 y_1^*)) = P^- \exp(i(x_1 y_2^* - x_2 y_1^*)) \quad .$$

Hence expression (20) becomes

$$\begin{aligned} h^+(\underline{x}) &= g(\underline{x}) \ P^+ \ \exp(-i(x_1 y_2^* - x_2 y_1^*)) + g(\underline{x}) \ P^- \ \exp(i(x_1 y_2^* - x_2 y_1^*)) \\ &= P^+ \ h(x_2, -x_1) + P^- \ h(-x_2, x_1) \\ &= P^+ \ h(-e_{12}\underline{x}) + P^- \ h(e_{12}\underline{x}) \end{aligned} \tag{21}$$

where

$$h(x_1, x_2) = g(\underline{x}) \ \exp(i(y_1^* x_1 + y_2^* x_2))$$

is the classical two-dimensional Gabor filter in case of a symmetric Gaussian.
Similarly we find

$$h^-(\underline{x}) = P^+ h(e_{12}\underline{x}) + P^- h(-e_{12}\underline{x}) \quad .$$

Remark 5.1 *The transformations $\underline{x} \to e_{12}\ \underline{x}$ and $\underline{x} \to -e_{12}\ \underline{x}$ represent an anti-clockwise, respectively a clockwise, rotation over a right angle.*

Naturally the Clifford-Gabor filters $\mathcal{G}^\pm$ can also be expressed in terms of classical one-dimensional Gabor filters which we now, for the sake of clarity, denote with a subindex specifying the angular frequency:

$$h^{y^*}(x) = g(x) \exp(iy^* x) \quad .$$

From (21) we immediately obtain

$$h^+(\underline{x}) = P^+ \ h^{-y_2^*}(x_1) \ h^{y_1^*}(x_2) + P^- \ h^{y_2^*}(x_1) \ h^{-y_1^*}(x_2) \quad .$$

A similar result holds for the impulse response of $\mathcal{G}^-$.

Finally, let us look for a relationship between the Clifford-Gabor filters $\mathcal{G}^{\pm}$ and the Gabor filters of Ebling and Scheuermann in case of a symmetric Gaussian (see subsection 5.2.2).
We have

$$\begin{aligned} h^e(\underline{x}) &= g(\underline{x}) \exp\left(e_{12}(y_1^* x_1 + y_2^* x_2)\right) \\ &= h^{\pm}(\pm x_2, \mp x_1) \\ &= h^{\pm}(\mp e_{12}\underline{x}) \ , \end{aligned}$$

taking into account that under the isomorphism between the Clifford algebras $\mathbb{R}_{2,0}$ and $\mathbb{R}_{0,2}$, both pseudoscalars e_{12} are isomorphic images of each other.

5.3.3 Localization in the spatial and in the frequency domain

An often cited property of Gabor filters is their optimal simultaneous localization in the spatial and the frequency domain. This makes them suitable for local frequency analysis. The notion "optimal simultaneous localization" is formalized by the *uncertainty principle*, which in its most cited form states that a nonzero function and its Fourier transform cannot both be sharply localized.
The uncertainty principle appeared in 1927 under the name "*Heisenberg inequality*" in the field of quantum mechanics in Heisenberg's paper [20].
However, it also has a useful interpretation in classical physics, namely it expresses a limitation on the extent to which a signal can be both time-limited and band-limited. This aspect of the uncertainty principle was already expounded by Wiener in a lecture in Göttingen in 1925. Unfortunately, no written record of this lecture seems to have survived, apart from the non-technical account in Wiener's autobiography [21]. The uncertainty principle became really fundamental in the field of signal processing after the publication of Gabor's famous article [10].

For a one-dimensional complex-valued signal f the uncertainty principle takes the form:

$$\Delta x \, \Delta y \geq \frac{1}{2} \ . \tag{22}$$

Here Δx denotes the *width* or *spatial uncertainty* of f, defined as the square root of the variance of the energy distribution of f:

$$(\Delta x)^2 = \frac{\int_{-\infty}^{+\infty} x^2 f(x) \, f^c(x) \, dx}{\int_{-\infty}^{+\infty} f(x) \, f^c(x) \, dx} \ .$$

Analogously, the *bandwidth* Δy is given by

$$(\Delta y)^2 = \frac{\int_{-\infty}^{+\infty} y^2 \, \mathcal{F}[f](y) \, \mathcal{F}^c[f](y) \, dy}{\int_{-\infty}^{+\infty} \mathcal{F}[f](y) \, \mathcal{F}^c[f](y) \, dy} \ .$$

The functions that minimize the inequality (22) are the complex Gabor filters. Hence, depending on the parameter σ, the Gabor filters are better localized in position or frequency space but they always exhibit the best possible joint localization.

Daugman extended the uncertainty principle to two-dimensional complex-valued filters or signals (see [22]):

$$\Delta x_1 \, \Delta x_2 \, \Delta y_1 \, \Delta y_2 \geq \frac{1}{4} \ , \tag{23}$$

where Δx_1 is defined by

$$(\Delta x_1)^2 = \frac{\int_{\mathbb{R}^2} x_1^2 \, f(x_1, x_2) \, f^c(x_1, x_2) \, dx_1 dx_2}{\int_{\mathbb{R}^2} f(x_1, x_2) \, f^c(x_1, x_2) \, dx_1 dx_2} \quad .$$

The uncertainties Δx_2, Δy_1 and Δy_2 are defined analogously.
It can be shown that two-dimensional complex Gabor filters achieve the minimum product of uncertainties, i.e.

$$\Delta x_1 \, \Delta x_2 \, \Delta y_1 \, \Delta y_2 = \frac{1}{4} \quad .$$

Let us now consider two-dimensional Clifford algebra-valued functions:

$$\begin{array}{rcl} f : \mathbb{R}^2 & \longrightarrow & \mathbb{C}_2 \\ \underline{x} = (x_1, x_2) & \longrightarrow & f(\underline{x}) = f(x_1, x_2) = f_0(\underline{x}) + f_1(\underline{x}) e_1 + f_2(\underline{x}) e_2 + f_{12}(\underline{x}) e_{12} \end{array}$$

with

$$f_i : \mathbb{R}^2 \longrightarrow \mathbb{C} \ \ , \ \ i = 0, 1, 2, 12 \ \ .$$

First we extend the definition of the uncertainties to these Clifford algebra-valued functions:

$$(\Delta x_1)^2 = \frac{\int_{\mathbb{R}^2} x_1^2 \, [f(\underline{x}) \, f^\dagger(\underline{x})]_0 \, dV(\underline{x})}{\int_{\mathbb{R}^2} [f(\underline{x}) \, f^\dagger(\underline{x})]_0 \, dV(\underline{x})}$$

and

$$(\Delta y_1)^2 = \frac{\int_{\mathbb{R}^2} y_1^2 \, [\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{y}) \, \left(\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{y})\right)^\dagger]_0 \, dV(\underline{y})}{\int_{\mathbb{R}^2} [\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{y}) \, \left(\mathcal{F}_{\mathcal{H}^\pm}[f](\underline{y})\right)^\dagger]_0 \, dV(\underline{y})} \quad .$$

Analogous definitions hold for Δx_2 and Δy_2.
For complex-valued signals, the real-valued energy distribution is given by $|f|^2 = f f^c$. For Clifford algebra-valued signals, this is given by

$$|f(\underline{x})|^2 = [f(\underline{x}) \, f^\dagger(\underline{x})]_0 = |f_0(\underline{x})|^2 + |f_1(\underline{x})|^2 + |f_2(\underline{x})|^2 + |f_{12}(\underline{x})|^2 \quad .$$

Hence, the uncertainty relation for two-dimensional Clifford algebra-valued signals is identical to Daugman's relation (23).

In case of the Clifford-Gabor filters $\mathcal{G}^\pm$ we have

$$\begin{aligned} |h^\pm(\underline{x})|^2 &= \Big[g(\underline{x}) \, \exp\left(\pm(\underline{x} \wedge \underline{y}^*)\right) g(\underline{x}) \, \exp\left(\mp(\underline{x} \wedge \underline{y}^*)\right) \Big]_0 \\ &= [(g(\underline{x}))^2]_0 \\ &= \frac{1}{4\pi^2 \sigma^4} \exp\left(-\frac{|\underline{x}|^2}{\sigma^2} \right) \ \ , \end{aligned}$$

where we have used the fact that

$$\left(\exp\left(\pm(\underline{x} \wedge \underline{y}^*)\right)\right)^\dagger = \exp\left(\mp(\underline{x} \wedge \underline{y}^*)\right) \quad .$$

Hence, for $\mathcal{G}^\pm$ we obtain

$$(\Delta x_1)^2 = \frac{\int_{\mathbb{R}^2} x_1^2 \, \exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) dV(\underline{x})}{\int_{\mathbb{R}^2} \exp\left(-\frac{|\underline{x}|^2}{\sigma^2}\right) dV(\underline{x})} = \frac{\sigma^2}{2} \quad .$$

Furthermore, we have

$$\left|\mathcal{F}_{\mathcal{H}^{\pm}}[h^{\pm}](\underline{y})\right|^2 = \frac{1}{4\pi^2} \exp\left(-\sigma^2 |\underline{y} - \underline{y}^*|^2\right) \ ,$$

which yields

$$(\Delta y_1)^2 = \frac{\int_{\mathbb{R}^2} y_1^2 \exp\left(-\sigma^2|\underline{y} - \underline{y}^*|^2\right) dV(\underline{y})}{\int_{\mathbb{R}^2} \exp\left(-\sigma^2|\underline{y} - \underline{y}^*|^2\right) dV(\underline{y})} = \frac{1}{2\sigma^2} \ .$$

Summarizing, the uncertainties of the Clifford-Gabor filters $\mathcal{G}^{\pm}$ are given by

$$\Delta x_1 = \Delta x_2 = \frac{\sigma}{\sqrt{2}} \quad \text{and} \quad \Delta y_1 = \Delta y_2 = \frac{1}{\sqrt{2}\sigma} \ ,$$

which implies

$$\Delta x_1 \ \Delta x_2 \ \Delta y_1 \ \Delta y_2 = \frac{1}{4} \ .$$

Hence, as in the classical setting, the Clifford-Gabor filters $\mathcal{G}^{\pm}$ are jointly optimally localized in the spatial and in the frequency domain.

6 RELATIONSHIP BETWEEN DERIVATIVES OF GAUSSIANS AND GABOR FILTERS

In [8] Michaelis investigates to which extent one-dimensional derivatives of Gaussians can be approximated by real Gabor filters.

Let us consider the derivatives of the Gaussian (12) with $a = \dfrac{1}{\sigma_h^2}$:

$$D_n(x) = (-1)^n N \sigma_h^{-n} \ H_n\left(\frac{x}{\sigma_h}\right) \ \exp\left(-\frac{x^2}{\sigma_h^2}\right) \ .$$

Note that they coincide, up to a constant, with the Hermite filters of subsection 3.1.

The asymptotic behavior of these Gaussian derivatives for $n \to \infty$ follows from the asymptotic behavior of the Hermite polynomials (see e.g. [23]):

$$\begin{aligned} H_{2s}(x) &= (-1)^s 2^s (2s-1)!! \ \exp\left(\frac{x^2}{2}\right) \left[\cos\left(\sqrt{4s+1}\, x\right) + \mathcal{O}(s^{-1/4})\right] \\ H_{2s+1}(x) &= (-1)^s 2^{s+1/2} (2s-1)!! \ \sqrt{2s+1} \ \exp\left(\frac{x^2}{2}\right) \left[\sin\left(\sqrt{4s+3}\, x\right) + \mathcal{O}(s^{-1/4})\right] \end{aligned}$$

with $(2s-1)!! = 1.3 \ldots (2s-1)$.

The above result implies the following asymptotic behavior of the Gaussian derivatives for $n \to \infty$:

$$D_n \approx \begin{cases} \exp\left(-\frac{x^2}{2\sigma_h^2}\right) \ \cos\left(\sqrt{2n+1} \ \frac{x}{\sigma_h}\right) & \text{for } n \text{ even} \\ \exp\left(-\frac{x^2}{2\sigma_h^2}\right) \ \sin\left(\sqrt{2n+1} \ \frac{x}{\sigma_h}\right) & \text{for } n \text{ odd.} \end{cases}$$

Recall that the one-dimensional real Gabor filters take the following form:

$$\begin{aligned} h_c(x) &= N \ \exp\left(-\frac{x^2}{2\sigma_g^2}\right) \ \cos\left(\frac{cx}{\sigma_g}\right) \\ h_s(x) &= N \ \exp\left(-\frac{x^2}{2\sigma_g^2}\right) \ \sin\left(\frac{cx}{\sigma_g}\right) \ . \end{aligned}$$

Hence, for large order n the derivatives of Gaussians look like Gabor filters. For n odd they look like Gabor sine filters, for n even they look like Gabor cosine filters. The Gabor filters have the same scale as the Gaussian derivatives, i.e. $\sigma_h = \sigma_g$, and the oscillation parameter of the Gabor filters is $c = \sqrt{2n+1}$.

In [8] Michaelis examines the quality of the fit between the Gaussian derivatives D_n with $a = \frac{1}{2\sigma^2}$ and real Gabor filters for the lower orders $n = 1, 2, 3$. It appears that D_1 can be approximated perfectly by Gabor filters. The worst fit is for D_2 where the L_2 error is 10%. For D_3 the error is 6%.

The comparison between the two-dimensional Clifford-Gabor and Clifford-Hermite filters is a topic of current research.

REFERENCES

[1] F. Brackx, N. De Schepper and F. Sommen, The Higher Dimensional Hermite Transform: A New Approach. *Complex Variables: Theory Appl.*, **48(3)**, 189-210, 2003.

[2] F. Brackx, N. De Schepper and F. Sommen, The Two-Dimensional Clifford-Fourier Transform. Accepted for publication in *Journal of Mathematical Imaging and Vision*.

[3] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*. Pitman Publ., Boston-London-Melbourne, 1982.

[4] R. Delanghe, F. Sommen and V. Souček, *Clifford Algebra and Spinor-Valued Functions*. Kluwer Acad. Publ., Dordrecht, 1992.

[5] J. B. Martens, The Hermite transform-theory. *IEEE Trans. Acoust. Speech, Signal Processing*, **ASSP-38(9)**, 1595-1606, 1990.

[6] F. Sommen, Special functions in Clifford analysis and axial symmetry. *JMAA*, **130(1)**, 110-133, 1988.

[7] F. Brackx, N. De Schepper and F. Sommen, The Hermite Transform in Quaternionic Analysis, *(digital) Proceedings 16th International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering*, 10-12 june 2003, Bauhaus-Universität Weimar.

[8] M. Michaelis, *Low level image processing using steerable filters*. PhD Thesis, Christian-Albrechts-Universität, Institut für Informatik und Praktische Mathematik, Kiel, 1995.

[9] I. N. Bronstein and K. A. Semendjajew, *Taschenbuch der Mathematik*. Harri Deutsch, Thun, 1985.

[10] D. Gabor, Theory of Communication. *J. Inst. Electr. Eng.*, **93**, 429-457, 1946.

[11] T. Bülow, *Hypercomplex Spectral Signal Representations for the Processing and Analysis of Images*. PhD Thesis, Christian-Albrechts-Universität, Institut für Informatik und Praktische Mathematik, Kiel, 1999.

[12] T. P. Weldon, W. E. Higgins and D. F. Dunn, Efficient Gabor Filter Design for Texture Segmentation. *Pattern Recognition*, **29(12)**, 2005-2015, December 1996.

[13] C. J. Rivero-Moreno and S. Bres, Conditions of Similarity between Hermite and Gabor Filters as Models of the Human Visual System. In: *Computer analysis of images and patterns*, Lecture Notes in Computer Science **LNCS 2756**, Springer-Verlag Berlin Heidelberg, 762-769, 2003.

[14] C. J. Rivero-Moreno and S. Bres, Les Filtres de Hermite et de Gabor Donnent-ils des Modèles Equivalents du Système Visuel Humain? In: *ORASIS'03, Journées Francophones des Jeunes Chercheurs en Vision par Ordinateur*, Gérardmer, France, 423-432, 2003.

[15] T. Bülow and G. Sommer, The hypercomplex signal - a novel approach to the multidimensional analytic signal. *IEEE Transactions on Signal Processing*, **49(11)**, 2844-2852, November 2001.

[16] T. Bülow and G. Sommer, Quaternionic Gabor filters for local structure classification. In: *14th International Conference on Pattern Recognition*, Brisbane, Australia, 1998, 808-810.

[17] J. Ebling and G. Scheuermann, Clifford Fourier Transform on Vector Fields. *IEEE Transactions on Visualization and Computer Graphics*, **11(4)**, IEEE Computer Society, 469-479, July/August 2005.

[18] J. Ebling and G. Scheuermann, Pattern Matching on Vector Fields using Gabor Filter. In: *Proceedings of the 14th IASTED International Conference on Visualization, Imaging and Image Processing*, Marbella, Spain, September 6-8, 2004, 825-830.

[19] F. Brackx, N. De Schepper and F. Sommen, The Clifford-Fourier Transform. *The Journal of Fourier Analysis and Applications*, **11(6)**, 669-681, 2005.

[20] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematic und Mechanik. *Zeit. Physik*, **43**, 172-198, 1927.

[21] N. Wiener, *I Am a Mathematician*. MIT Press, Cambridge, 1956.

[22] J. G. Daugman, Uncertainty relation for resolution in space, spatial frequency, and orientation by two-dimensional visual cortical filters. *Journal Opt. Soc. Am. A*, **2(7)**, 1160-1169, 1985.

[23] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*. Academic Press, New York, USA, 1980.